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Phase Retrieval and Norm Retrieval in Unitary Systems

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Abstract: In a separable Hilbert space $\mathcal{H}$, we introduce phase and norm retrievable frame generators for a unitary system $\mathcal{U}$ and show that phase retrievable and norm retrievable vectors and subspaces can be obtained with a unitary system structure. We give the conditions under which a complete wandering subspace W for a unitary system $\mathcal{U}$ can generate vectors and subspaces that do phase retrieval and norm retrieval in $\mathcal{H}$.

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1.Introduction

In Hilbert spaces, frames are a generalization of Riesz bases that provide representations of vectors that are not always unique. Because of the redundancy of frames, they are highly valuable in a wide range of application areas which includes wavelet and frequency analysis, signal and image processing, and many other areas of mathematics (Daubechies, I. (1992), Feichtinger, H. G., and Strohmer, T. (Eds.). (2012), Candès, E. J., and Donoho, D. L. (2004)). Recently, many researchers have focused their attention on phase and norm retrievable frames in Hilbert spaces which are first introduced by (Balan, R., Casazza, P. and Edidin, D. (2006), Bahmanpour, S., Cahill, J., Casazza, P. G., Jasper, J., and Woodland, L. M. (2014)). Phase retrieval allows us to reconstruct a signal from the magnitudes of its linear measurements which makes them widely used in the literature.

Wandering vectors and subspaces for unitary and isometry systems have been studied in (Dai, X., and Larson, D. R. (1998), Han, D. (1998), Robertson, J. B. (1965), Halmos, P. R. (1961)). It was shown by Dai and Larson in (Dai, X., and Larson, D. R. (1998)) that orthogonal wavelets for dilation-translation unitary systems can be thought of as wandering vectors. In Hilbert spaces, Goodman, Lee, and Tang in (Goodman, T. N. T., Lee, S. L., and Tang, W. S. (1993) give the relations between multiresolution analysis and wandering subspaces for unitary operators which is important in wavelets theory with its practical applications in image and signal processing.

In this paper, we study phase retrivable and norm retrievable frames in Hilbert spaces, which are generated with a unitary system structure. We show that if there exists a complete wandering subspace for a unitary system, then it is possible to have phase retrivable and norm retrievable frames in Hilbert spaces.

We begin by providing the background information required for the article. We refer readers for more details about frame theory (Christensen, O. (2003), Casazza, P. G., Kutyniok, G., and Philipp, F. (2013)), phase and norm retrievable frames (Balan, R. et al. (2006), Bahmanpour, S., Cahill, J., Casazza, P. G., Jasper, J., and Woodland, L. M. (2014), Botelho-Andrade, S., Casazza, P. G., Van Nguyen, H., and Tremain, J. C. (2016), Casazza, P. G., Ghoreishi, D., Jose, S., and Tremain, J. C. (2017), Casazza, P. G., and Kutyniok, G. (2004)), and wandering subspaces for unitary systems (Bhattacharjee, M., Eschmeier, J., Keshari, D. K., and Sarkar, J. (2017), Liu, A., and Li, P. (2016), Han, D., Larson, D. R., Papadakis, M., and Stavropoulos, T. (1999))

Definition 1.1. [Duffin, R. J., and Schaeffer, A. C. (1952)] A collection of vectors $\{x_i\}_{i\in I}$ in a separable Hilpert space $\mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A \parallel x \parallel^2 \leq \sum_{i\in I} |\langle x, x_i\rangle|^2 \leq B \parallel x \parallel^2 \text{, for all } x \in \mathcal{H}.$$

A and B are called lower and upper frame bounds, respectively.

Given a frame $\mathcal{F} = \{x_i\}_{i\in I}$ in $\mathcal{H}$. Let $\{e_i\}_{i\in I}$ be a standard orthonormal basis for $\ell^2(I)$. The analysis operator $\Phi: \mathcal{H} \to \ell^2(I)$ of $\mathcal{F}$ is defined by

$$\Phi(x) = \sum_{i\in I} \langle x, x_i\rangle e_i \text{ for all } x \in \mathcal{H}.$$

The synthesis operator of $\mathcal{F}$ is the adjoint Φ^* of the analysis operator Φ which is given by

$$\Phi^*: \ell^2(I) \to \mathcal{H},\ \Phi^*((c_i)_{i\in I}) = \sum_{i\in I} c_i x_i.$$

The frame operator $S = \Phi^*\Phi: \mathcal{H} \to \mathcal{H}$ of the frame $\mathcal{F}$ is defined by

$$S(x) = \Phi^*\Phi(x) = \sum_{i\in I} \langle x, x_i\rangle x_i$$

Given a frame F, the frame operator S satisfies the inequality $AI \leq S \leq BI$, where A and B are upper and lower frame bounds and I is the identity operator on $\mathcal{H}$. The frame operator S is a bounded, invertible, and positive operator.

Now, we are ready to define frame subspaces in separable Hilbert Spaces.

Definition 1.2. [Casazza, P. G., and Kutyniok, G. (2004)**]** Let $\{W_i\}_{i\in I}$ be a collection of closed subspaces in a Hilbert space $\mathcal{H}$ and P_i be the orthogonal projections from $\mathcal{H}$ onto W_i for each $i \in I$. The collection of $\{W_i\}_{i\in I}$ is a frame subspace with respect to weights $\{c_i\}_{i\in I}$, $c_i > 0$ for all $i \in I$, in $\mathcal{H}$, if there exist A and B with $0 < A \leq B < \infty$ such that

$$A \parallel x \parallel^2 \leq \sum_{i \in I} c_i^{\,2} \|P_i x\|^2 \leq B \parallel x \parallel^2 \text{, for all } x \in \mathcal{H}. \qquad (1.1)$$

A and B are called upper and lower frame bounds, respectively. The frame of subspaces $\{W_i\}_{i \in I}$ with respect to weights $\{c_i\}_{i \in I}$ is called tight if there exist constants A and B such that $A = B$ and satisfies the inequality in (1.1).

Definition 1.3 [Balan et al. 2006] A collection of vectors $\{x_i\}_{i \in I}$ in $\mathcal{H}$ is called phase retrieval if for all $x, y \in \mathcal{H}$ which satisfies $|\langle x, x_i \rangle| = |\langle y, x_i \rangle|$ for all $i \in I$, then $x = cy$ where $c = \pm 1$ in $\mathcal{H}$.

Definition 1.4 [Bahmanpour et al. 2014] A collection of vectors $\{x_i\}_{i \in I}$ in $\mathcal{H}$ is called norm retrieval if for all $x, y \in \mathcal{H}$ which satisfies $|\langle x, x_i \rangle| = |\langle y, x_i \rangle|$ for all $i \in I$, then $\parallel x \parallel = \parallel y \parallel$.

Definition 1.5 [Bahmanpour et al. 2014] Let $\{W_i\}_{i \in I}$ be a collection of subspaces in $\mathcal{H}$ and define $\{P_i\}_{i \in I}$ to be the orthogonal projections onto each of these subspaces. We say that $\{W_i\}_{i \in I}$ (or $\{P_i\}_{i \in I}$) yields phase retrieval if for $x, y \in \mathcal{H}$ satisfying $\|P_i x\| = \|P_i y\|$ for all $i \in I$, then $x = cy$ for some scalar c with $c = \pm 1$.

Definition 1.6 [Bahmanpour et al. 2014] Let $\{W_i\}_{i \in I}$ be a collection of subspaces in $\mathcal{H}$ and define $\{P_i\}_{i \in I}$ to be the orthogonal projections onto each of these subspaces. We say that $\{W_i\}_{i \in I}$ (or $\{P_i\}_{i \in I}$) yields norm retrieval if for $x, y \in \mathcal{H}$ satisfying $\|P_i x\| = \|P_i y\|$ for all $i \in I$, then $\parallel x \parallel = \parallel y \parallel$.

2. Wandering Subspaces and Unitary Systems

Given an operator T on a Hilbert space $\mathcal{H}$, a closed subspace $W \subset \mathcal{H}$ is called a wandering subspace for T, which is defined by (Halmos,1961), if W is orthogonal to the power of all its images under T. That is

$$W \perp T^k W, \ (k \geq 1).$$

W is called a generating wandering subspace for an operator T if $\mathcal{H} = \overline{\text{span}}\{T^k W : k \geq 0\}$.

It is shown in (Halmos, 1961) that for isometries V, a wandering subspace W for V satisfies

$$V^j W \perp V^k W, \text{for all } k \neq j$$

where j and k are non-negative integers. When U is a unitary, wandering subspaces satisfies

$$U^j W \perp U^k W, \ (k \neq j)$$

where j and k are possibly non-negative integers.

A unitary system $\mathcal{U}$, which is defined by Dai and Larson (Dai et al., 1998), is a subset of unitary operators acting on $\mathcal{H}$ that includes the identity operator I. Let W be a closed subspace of $\mathcal{H}$.

W is called a wandering subspace for a unitary system $\mathcal{U}$ if UW and VW are orthogonal for any non-equal unitaries $U, V \in \mathcal{U}$. W is called a complete wandering subspace for a unitary system $\mathcal{U}$ if we have, for any $U, V \in \mathcal{U}$ such that $U \neq V$,

$$\text{U}W \perp \text{V}W \text{ and } \mathcal{H} = \overline{span}\{UW : U \in \mathcal{U}\}.$$

Let $\{x_i\}_{i \in I}$ be an orthonormal basis for W. Then, the set of $\{Ux_i\}_{i \in I}$ is also an orthonormal basis for UW since any $U \in \mathcal{U}$ is a unitary operator. This implies that W is a complete wandering subspace for a unitary system $\mathcal{U}$ if and only if $\{Uxi : U \in \mathcal{U}, i \in I\}$ is an orthonormal basis for $\mathcal{H}$.

$$C_W(\mathcal{U}) = \{T \in B(\mathcal{H}): (TU - UT)W = \{O\}\, for\; U \in \mathcal{U}\}$$

is called the local commutant of $\mathcal{U}$ at W. $(TU - UT)W = \{O\}$ implies that $TUW = UTW$.

3. Phase Retrieval and Norm retrieval in Unitary Systems

Authors (Liu et al., 2016) show that wandering subspaces can be fusion frame generators for unitary systems. In this section, we investigate phase and norm retrievable frames, which have a unitary system structure.

Definition 3.1. Let $\mathcal{U}$ be a unitary system on a Hilbert space $\mathcal{H}$. A closed subspace W of $\mathcal{H}$ is called a phase retrievable frame generator for $\mathcal{U}$ if the collection of subsets $\{\mathrm{U}W\}_{U\in\mathcal{U}}$ is phase retrievable in $\mathcal{H}$.

Definition 3.2 Let $\mathcal{U}$ be a unitary system on a Hilbert space $\mathcal{H}$. A closed subspace W of $\mathcal{H}$ is called a norm retrievable frame generator for $\mathcal{U}$ if the collection of subsets $\{\mathrm{U}W\}_{U\in\mathcal{U}}$ is norm retrievable in $\mathcal{H}$.

Theorem 3.3 Let $\mathcal{U}$ be a unitary system on a Hilbert space $\mathcal{H}$ and $W \subset \mathcal{H}$ be a phase retrievable frame generator for $\mathcal{U}$. If T is an invertible operator in $C_W(\mathcal{U})$, then TW is also a phase retrievable frame generator for $\mathcal{U}$.

Proof: Suppose W is a phase retrievable frame generator for a unitary system $\mathcal{U}$ on $\mathcal{H}$. Given an invertible operator T in $C_W(\mathcal{U})$, we want to show that TW is a phase retrievable frame generator for $\mathcal{U}$. To do this, we need to prove that the collection of subsets $\{\mathrm{UT}W\}_{U\in\mathcal{U}}$ is phase retrievable in $\mathcal{H}$.

Let $\{x_i\}_{i\in I}$ be a set of orthonormal basis in W. For any unitary operator $U \in \mathcal{U}$, the set of vectors $\{Ux_i\}_{i\in I}$ is an orthonormal basis in UW. Let P_U be the orthogonal projection from $\mathcal{H}$ onto the subspace UW. For any $x \in \mathcal{H}$, we have

$$P_u(x) = \sum_{i\in I}\langle x, Ux_i\rangle Ux_i.$$

Since $T \in C_W(\mathcal{U})$, we can write $TUW = UTW$ for all $U \in \mathcal{U}$. Let $\{P_U\}_{U\in\mathcal{U}}$ be the orthogonal projections onto the subspaces $\{\mathrm{U}W\}_{U\in\mathcal{U}}$. The orthogonal projections from $\mathcal{H}$ onto the subspaces $\{\mathrm{TU}W\}_{U\in\mathcal{U}}$ are $\{Q_U = TP_UT^*\}_{U\in\mathcal{U}}$. For any $x \in \mathcal{H}$, we can write

$$\begin{aligned}
\|Q_U(x)\|^2 &= \langle Q_U(x), Q_U(x)\rangle \\
&= \langle Q_U(x), x\rangle \\
&= \langle TP_UT^*(x), x\rangle \\
&= \langle P_UT^*(x), T^*(x)\rangle \\
&= \langle \sum_{i\in I}\langle T^*x, Ux_i\rangle Ux_i\,, T^*(x)\rangle \\
&= \sum_{i\in I}|\langle T^*x, Ux_i\rangle|^2 \\
&= \|P_UT^*(x)\|^2.
\end{aligned}$$

Hence, given any $x, y \in \mathcal{H}$ which satisfies $\|Q_U(x)\| = \|Q_U(y)\|$ for all $U \in \mathcal{U}$ implies that $\|P_UT^*(x)\| = \|P_UT^*(y)\|$ for all $U \in \mathcal{U}$. Since W is a phase retrievable frame generator for $\mathcal{U}$ and T is an invertible operator, $\|P_UT^*(x)\| = \|P_UT^*(y)\|$ for all $U \in \mathcal{U}$ states that $T^*(x) = \pm T^*(y)$ and $x = \pm y$. This says that TW is a phase retrievable frame generator for $\mathcal{U}$.

Theorem 3.4 Let $\mathcal{U}$ be a unitary system on a Hilbert space $\mathcal{H}$. Suppose W_1 and W_2 are two complete wandering subspaces for $\mathcal{U}$ such that $dimW_1 = dimW_2$. Then, W_1 is a phase retrievable frame generator for $\mathcal{U}$ if and only if W_2 is a phase retrievable frame generator for $\mathcal{U}$.

Proof. Let W_1 and W_2 be two complete wandering subspaces for a unitary system $\mathcal{U}$ in a Hilbert space $\mathcal{H}$ such that $dimW_1 = dimW_2$. Suppose, W_1 is a phase retrievable frame generator for $\mathcal{U}$. Let $\{x_i\}_{i\in I}$ and $\{y_i\}_{i\in I}$ be orthonormal bases for W_1 and W_2, respectively, where the cardinality of the index set I is equal to $dimW_1$. Then, both $\{Uxi: U \in \mathcal{U}, i \in I\}$ and $\{Uyi: U \in \mathcal{U}, i \in I\}$ are orthonormal bases for $\mathcal{H}$ by the definition of the complete wandering subspaces for a unitary system $\mathcal{U}$.

Let us define a unitary operator T on $\mathcal{H}$ such that $TUx_i = Uy_i$ for all $i \in I$ and $U \in \mathcal{U}$. It is obvious that $TW_1 = W_2$. For any $U \in \mathcal{U}$, since the identity operator $I_d \in \mathcal{U}$, we have

$$TUx_i = Uy_i = UTx_i \text{ for all } i \in I.$$

Hence, $T \in C_W(\mathcal{U})$ and by the Theorem 3.3., $W_2$is a phase retrievable frame generator for $\mathcal{U}$. A similar calculation shows that if $W_2$is a phase retrievable frame generator for $\mathcal{U}$, then $W_1$is also a phase retrievable frame generator for $\mathcal{U}$.

Theorem 3.5. Let $\mathcal{U}$ be a unitary system on a Hilbert space $\mathcal{H}$ and $W_1 \subset \mathcal{H}$ is a complete wandering subspace for $\mathcal{U}$. Let $W_2 \subset \mathcal{H}$ be a closed subspace such that $dimW_1 = dimW_2$. If there exists a coisometry $T \in C_{W_1}(\mathcal{U})$ such that $TW_1 = W_2$ and T is isometric on W_1, then W_2 is a norm retrievable frame generator for $\mathcal{U}$.

Proof. Suppose $T \in C_{W_1}(\mathcal{U})$ is a coisometry such that $TW_1 = W_2$ and T is isometric on W_1. For an orthonormal basis $\{x_i\}_{i\in I}$ in W_1, let $Tx_i = y_i$ for all $i \in I$. Since $TW_1 = W_2$ and T is isometric on W_1, the set of vectors $\{y_i\}_{i\in I}$ forms an orthonormal basis for W_2. Let P_U denotes the orthogonal projection from $\mathcal{H}$ onto the subspace $UW_1$and Q_U denotes the orthogonal projection from $\mathcal{H}$ onto the subspace UW_2 for each $U \in \mathcal{U}$.

$T \in C_{W_1}(\mathcal{U})$ implies that $TUx = UTx$ for all $x \in W_1$ and $U \in \mathcal{U}$. Since $\{Uxi: U \in \mathcal{U}, i \in I\}$ is an orthonormal basis for $\mathcal{H}$ and T^* is an isometry, for all $x \in \mathcal{H}$, we can write

$$\begin{aligned}
\| x \|^2 = \| T^*x \|^2 &= \sum_{i\in I, U\in\mathcal{U}} |\langle T^*x, Ux_i\rangle|^2 \\
&= \sum_{i\in I, U\in\mathcal{U}} |\langle x, TUx_i\rangle|^2 \\
&= \sum_{i\in I, U\in\mathcal{U}} |\langle x, UTx_i\rangle|^2 \\
&= \sum_{i\in I, U\in\mathcal{U}} |\langle x, Uy_i\rangle|^2 \\
&= \sum_{i\in I, U\in\mathcal{U}} |\langle Q_U x, Uy_i\rangle|^2 \\
&= \sum_{U\in\mathcal{U}} \|Q_U x\|^2.
\end{aligned}$$

This implies that the set of projections $\{\mathrm{U}W_2\}_{U\in\mathcal{U}}$ is a norm retrievable frame for $\mathcal{H}$ and W_2 is a norm retrievable frame generator for $\mathcal{U}$.

Proposition 3.6. Let $W \subset \mathcal{H}$ be a generating wandering subspace for an isometry T. Then, the set of subspaces $\{T^kW\}_{k\geq 0}$ is norm retrievable in $\mathcal{H}$.

Proof. Let W be a generating wandering subspace for an isometry T and P_k be the orthogonal projection onto the subspace T^kW for each $k \geq 0$. Since W is a generating wandering subspace for T, we have $T^{k_1}W \perp T^{k_2}W$ for all $k_1 \neq k_2$ and $\mathcal{H} = \overline{\text{span}}\{T^kW: k \geq 0\}$.

Hence, for any $x \in \mathcal{H}$, we can write $x = \sum_{k\geq 0} P_k(x)$. For any $k_1, k_2 \geq 0$, if $k_1 \neq k_2$, then $P_{k_1}P_{k_2} = 0$. That is, for any $x, y \in \mathcal{H}$ satisfying $\|P_kx\| = \|P_ky\|$ for all $k \geq 0$. We have

$$\| x \|^2 = \sum_{k\geq 0} \|P_k(x)\|^2 = \sum_{k\geq 0} \|P_k(y)\|^2 = \| y \|^2.$$

Which says that the set of subspaces $\{T^kW\}_{k\geq 0}$ is norm retrievable in $\mathcal{H}$.

Proposition 3.7. Let $W \subset \mathcal{H}$ be generating wandering subspace for an invertible operator T such that $\dim W < \infty$. If the set of vectors $\{x_i\}_{i\in I}$ does phase retrieval in W, then the set of vectors $\{T^kx_i\}_{i\in I, k\geq 0}$ does norm retrieval in H.

Proof: Let T be an invertible operator and $W \subset \mathcal{H}$, $\dim W < \infty$, be a generating wandering subspace for T. Suppose the set of vectors $\{x_i\}_{i\in I}$ does phase retrieval in W. Since phase retrieval condition is preserved under the invertible operators as shown in (Thm. 4.8, Bahmanpour et al. 2014), the set of vectors $\{T^kx_i\}_{i\in I}$ does phase retrieval (hence, norm retrieval) in the subspace $T^kW \subset \mathcal{H}$ for each $k \geq 0$. Let P_k be the orthogonal projection onto the subspace T^kW for each $k \geq 0$.

Given $x, y \in \mathcal{H}$, assume we know $|\langle x, T^kx_i\rangle| = |\langle y, T^kx_i\rangle|$ for all $k \geq 0$ and $i \in I$. For each fixed k, we have $P_kT^kx_i = T^kx_i$. This allows us to write $|\langle P_kx, T^kx_i\rangle| = |\langle P_ky, T^kx_i\rangle|$ for all $i \in I$. Since the set of vectors $\{T^kx_i\}_{i\in I}$ does norm retrieval in T^kW, we have $\|P_kx\| = \|P_ky\|$ for each $k \geq 0$.

Since W is a generating wandering subspace for T, we have $\mathcal{H} = \overline{\text{span}}\{T^kW: k \geq 0\}$, and any $x \in \mathcal{H}$ can be written as $\mathrm{x} = \sum_{k\geq 0} \mathrm{c_k} P_k(x)$ and $\| x \|^2 = \sum_{k\geq 0} \mathrm{c_k}\|P_k(x)\|^2$. This allows us to conclude that the set of projections $\{P_k\}_{k\geq 0}$ does norm retrieval in $\mathcal{H}$ and $\|x\| = \|y\|$. Hence, the set of vectors $\{T^kx_i\}_{i\in I, k\geq 0}$ does norm retrieval in $\mathcal{H}$.

Proposition 3.8. Let $W \subset \mathcal{H}$ be a generating wandering subspace of a unitary operator T. If the set of vectors $\{x_i\}_{i\in I}$ does norm retrieval in W, then the set of vectors $\{T^kx_i\}_{i\in I, k\geq 0}$ does norm retrieval in $\mathcal{H}$.

Proof. This proposition has a similar proof to the Proposition 3.7.

Proposition 3.9. Let $W \subset \mathcal{H}$ be a finite dimensional generating wandering subspace for an invertible self-adjoint operator T. If $\{x_i\}_{i\in I}$ is a set of orthogonal vectors in W with the cardinality of the index set $I = dimW < \infty$, then the set of vectors $\{T^kx_i\}_{i\in I, k\geq 0}$ does norm retrieval in $\mathcal{H}$.

Proof. Suppose W is a generating wandering subspace for an invertible self-adjoint operator T with $dimW < \infty$ and $\{x_i\}_{i\in I}$ is a set of orthogonal vectors in W. For any $x_i, x_j \in W$, since we have $W \perp T^kW$ for all $k \geq 1$, we can write

$$\langle T^kx_i, T^kx_j\rangle = \langle x_i, (T^k)^*T^kx_j\rangle = \langle x_i, T^{2k}x_j\rangle = 0.$$

This states that the set of vectors $\{T^kx_i\}_{i\in I}$ is an orthogonal basis in T^kW for each $k \geq 1$. By normalizing each vector T^kx_i , we have an orthonormal basis $\left\{\frac{T^kx_i}{\|T^kx_i\|}\right\}_{i\in I}$ in $\mathcal{H}$ since $\mathcal{H} = \overline{\text{span}}\{T^kW: k \geq 0\}$. It is obvious from the definition of norm retrieval that the orthonormal basis does norm retrieval in $\mathcal{H}$. Hence, the set of vectors $\{T^kx_i\}_{i\in I, k\geq 0}$ does norm retrieval in $\mathcal{H}$.

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